FDPE Math Camp
2014

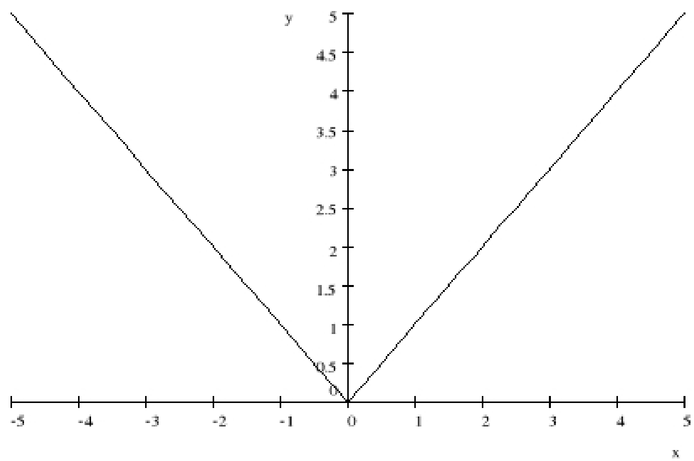
Exercise Set 2

Suggested Solutions

1. Continuity and differentiability of functions

   (a) Both of the functions here are continuous, but the first one is not differentiable.

      i. The function $f(x) = |x|$ is continuous but not differentiable.



As we can see from the figure, the graph of the function has no breaks, but a kink at $x = 0$. However, since

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^-} |x| = 0$$

$f(x)$ is continuous.

Secondly, I claim that $f(x)$ not differentiable at $x = 0$. To see this, consider two sequences which converge to zero

$$h_n = \{0.1, 0.01, ..., 0.1^n\}$$

and

$$k_n = \{-0.1, -0.01, ..., -0.1^n\}$$

Now recall the definition of derivative

$$\lim_{h_n \to 0} \frac{f(x_0 + h_n) - f(x_0)}{h_n}$$

Substituting and evaluating at $x_0 = 0$

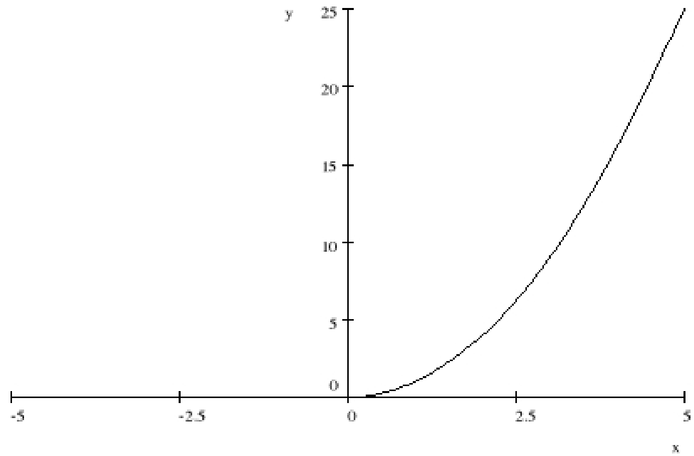
$$\lim_{h_n \to 0} \frac{f(0 + h_n) - f(0)}{h_n} = \frac{h_n - 0}{h_n} = 1$$

and

$$\lim_{k_n \to 0} \frac{f(0 + k_n) - f(0)}{k_n} = \frac{-k_n - 0}{k_n} = -1$$

      ii. $f(x) = \begin{cases} x^2 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$ is both continuous and differentiable.

Its derivative is $f'(x) = \begin{cases} 2x & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$ which is also continuous.



(b) Homogeneity of degree r means by definition

$$f(\lambda \mathbf{x}) = \lambda^r f(\mathbf{x})$$

Differentiating with respect to λ yields

$$\sum_{i=1}^{n} x_i \frac{\partial f(\lambda x_1, ..., \lambda x_n)}{\partial x_i} = r\lambda^{r-1} f(\mathbf{x})$$

Evaluating at $\lambda = 1$ yields

$$\sum_{i=1}^{n} x_i \frac{\partial f(\mathbf{x})}{\partial x_i} = rf(\mathbf{x})$$

2. Implicit Function Theorem

(a) In this example, we will consider a system of two linear equations and two unknowns so we will apply the linear implicit function theorem. Note that since the equations are linear in all variables, the partial derivatives will always obtain constant values.

i. Consider the function

$$\mathbf{F} = \left(\mathbf{F}^1, \mathbf{F}^2\right) = (2x_1 - 3x_2 + y_1 - y_2, x_1 + 2x_2 + y_1 - 2y_2) = \mathbf{c}$$

or

$$\mathbf{F}(x, y) = \mathbf{c}$$

Totally differentiating yields

$$\mathbf{F}_x d\mathbf{x} + \mathbf{F}_y d\mathbf{y} = \mathbf{0} \iff \mathbf{F}_y d\mathbf{y} = -\mathbf{F}_x d\mathbf{x}$$

where the matrices of partial derivatives are given by

$$\mathbf{F}_x = \begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix}$$

and

$$\mathbf{F}_y = \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix}$$

Recall that the implicit function theorem holds if the matrix $\mathbf{F}_y$ is nonsingular. Here,

$$\det \mathbf{F}_y = \begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix} = -2 + 1 \neq 0$$

Now since $\mathbf{F}_y$ is invertible, we can write

$$d\mathbf{y} = -\mathbf{F}_y^{-1}\mathbf{F}_x d\mathbf{x}$$

That is

$$d\mathbf{y} = -\begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix}^{-1} \begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix} d\mathbf{x}$$

$$= \begin{pmatrix} -2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix} d\mathbf{x} = = \begin{pmatrix} -3 & 8 \\ -1 & 5 \end{pmatrix} d\mathbf{x}$$

ii. Consider the function

$$\mathbf{F} = \left(\mathbf{F}^1, \mathbf{F}^2\right) = (2x_1 - x_2 + 2y_1 - y_2, 3x_1 + 2x_2 + y_1 + 2y_2).$$

Again, we define

$$\mathbf{F}_x = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$$

and

$$\mathbf{F}_y = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

The linear implicit function theorem holds since

$$\det \mathbf{F}_y = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 5$$

Similarly to (i), we can find the values of partial derivatives from

$$d\mathbf{y} = -\mathbf{F_y^{-1}F_x dx} = -\frac{1}{5}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix} d\mathbf{x} = \begin{pmatrix} -\frac{7}{5} & 0 \\ -\frac{4}{5} & -1 \end{pmatrix} d\mathbf{x}$$

(b) The equilibrium in a model is given by

$$f_1(y_1, y_2) = 3y_1y_2^2 - 2\alpha y_1 + \beta = 0$$

$$f_2(y_1, y_2) = 3y_1y_2 - 3y_2 + \alpha = 0$$

which is a nonlinear system of equations with two endogenous and two exogenous variables. In the matrix form

$$f(y; \alpha, \beta) = \begin{pmatrix} f_1(y; \alpha, \beta) \\ f_2(y; \alpha, \beta) \end{pmatrix} = \begin{pmatrix} 3y_1y_2^2 - 2\alpha y_1 + \beta \\ 3y_1y_2 - 3y_2 + \alpha \end{pmatrix}$$

Again, we calculate the matrices of partial derivatives

$$f_y = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix} = \begin{pmatrix} 3(y_2)^2 - 2\alpha & 6y_1y_2 \\ 3y_2 & 3y_1 - 3 \end{pmatrix}$$

and

$$f_{\alpha\beta} = \begin{pmatrix} \frac{\partial f_1}{\partial \alpha} & \frac{\partial f_1}{\partial \beta} \\ \frac{\partial f_2}{\partial \alpha} & \frac{\partial f_2}{\partial \beta} \end{pmatrix} = \begin{pmatrix} -2y_1 & 1 \\ 1 & 0 \end{pmatrix}$$

By implicit function theorem we have

$$\begin{pmatrix} dy_1 \\ dy_2 \end{pmatrix} = -f_y^{-1} f_{\alpha\beta} \begin{pmatrix} d\alpha \\ d\beta \end{pmatrix} = 0$$

$$\iff \begin{pmatrix} dy_1 \\ dy_2 \end{pmatrix} = -\begin{pmatrix} 3(y_2)^2 - 2\alpha & 6y_1y_2 \\ 3y_2 & 3y_1 - 3 \end{pmatrix}^{-1} \begin{pmatrix} -2y_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d\alpha \\ d\beta \end{pmatrix}$$

$$\Longleftrightarrow \begin{pmatrix} dy_1 \\ dy_2 \end{pmatrix} = \frac{1}{\left(3\left(y_2\right)^2 - 2\alpha\right)\left(3y_1 - 3\right) - 18y_1\left(y_2\right)^2} \begin{pmatrix} 3 - 3y_1 & 6y_1y_2 \\ 3y_2 & 2\alpha - 3\left(y_2\right)^2 \end{pmatrix} \begin{pmatrix} -2y_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d\alpha \\ d\beta \end{pmatrix}$$

$$\Longleftrightarrow \begin{pmatrix} dy_1 \\ dy_2 \end{pmatrix} = \frac{1}{9\left(y_2\right)^2\left(y_1 + 1\right) + 6\alpha\left(1 - 6y_1\right)} \begin{pmatrix} -6y_1\left(1 + y_1 - y_2\right) & 3 - 3y_1 \\ -6y_1y_2 + 2\alpha - 3\left(y_2\right)^2 & 3y_2 \end{pmatrix} \begin{pmatrix} d\alpha \\ d\beta \end{pmatrix}$$

Note that we must have $9\left(y_2\right)^2\left(y_1 + 1\right) + 6\alpha\left(1 - 6y_1\right) \neq 0$ for the implicit function theorem to be applicable!
The matrix of partial derivatives

$$\begin{pmatrix} \frac{\partial y_1}{\partial \alpha} & \frac{\partial y_1}{\partial \beta} \\ \frac{\partial y_2}{\partial \alpha} & \frac{\partial y_2}{\partial \beta} \end{pmatrix} = \frac{1}{9\left(y_2\right)^2\left(y_1 + 1\right) + 6\alpha\left(1 - 6y_1\right)} \begin{pmatrix} 6y_1\left(y_1 + y_2 - 1\right) & 3 - 3y_1 \\ -6y_1y_2 + 2\alpha - 3\left(y_2\right)^2 & 3y_2 \end{pmatrix}$$

Evaluating at the point $(y_1, y_2, \alpha, \beta) = (1, 1, 0, -3)$

$$\frac{1}{9(1+1) + 6 \cdot 0 (1 - 6 \cdot 1)} \begin{pmatrix} 6(1 + 1 - 1) & 3 - 3 \\ -6 + 2 \cdot 0 - 3 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{3} & 0 \\ -\frac{1}{2} & \frac{1}{6} \end{pmatrix}$$

3. Log-linearization

(a) Using the definitions from the lecture notes

$$XY = e^{\log X} e^{\log Y} = e^{x+y}$$

By using the multivariate log-linearization around (X^*Y^*), we yield an approximation for XY around the point X^*Y^*

$$XY \approx e^{x^*+y^*} + e^{x^*+y^*}(x - x^*) + e^{x^*+y^*}(y - y^*)$$

Substituting back $e^{x^*+y^*} = X^*Y^*$ and defining $\hat{x} = x - x^*$ and $\hat{y} = y - y^*$, we obtain

$$X^*Y^*(1 + \hat{x} + \hat{y})$$

(b) Now we have

$$e^{k_{t+1}} = Ae^{\alpha k_t} - e^{c_t}$$

log-linearizing both sides and substituting for $\bar{Y} = A\bar{K}^\alpha$ yields

$$\begin{aligned} \bar{K}\left(1 + k_{t+1} - \bar{k}\right) &\approx \left(A\bar{K}^\alpha - \bar{C}\right) + \alpha A\bar{K}^\alpha\left(k_t - \bar{k}\right) - \bar{C}\left(c_t - \bar{c}\right) \\ &= \bar{K} + \alpha\bar{Y}\left(k_t - \bar{k}\right) - \bar{C}\left(c_t - \bar{c}\right) \end{aligned}$$

Dividing both sides by $\bar{K}$ and rearranging

$$k_{t+1} - \bar{k} \approx \alpha\frac{\bar{Y}}{\bar{K}}\left(k_t - \bar{k}\right) - \frac{\bar{C}}{\bar{K}}\left(c_t - \bar{c}\right)$$

4. Nonlinear dynamics and local stability of steady states

$$\begin{cases} x_{t+1} = x_t y_t + y_t^2 - 1 \\ y_{t+1} = x_t + 3x_t^2 y_t - 3 \end{cases}$$

(a) To show that $(1, 1)$ is a steady state we can simply substitute $x = x_{t+1} = x_t = 1$ and $y = y_{t+1} = y_t = 1$ in the equations:

$$\begin{cases} 1 = 1 + 1 - 1 \\ 1 = 1 + 3 - 3 \end{cases}$$

which obviously holds.

(b) First-order Taylor approximation around $(1,1)$. We will need the matrix of first-order derivatives

$$\Delta = \begin{pmatrix} \frac{\partial x_{t+1}}{\partial x_t} & \frac{\partial x_{t+1}}{\partial y_t} \\ \frac{\partial y_{t+1}}{\partial x_t} & \frac{\partial y_{t+1}}{\partial y_t} \end{pmatrix} = \begin{pmatrix} y_t & x_t + 2y_t \\ 1 + 6x_t y_t & 3x_t^2 \end{pmatrix}$$

Evaluating at $(1,1)$ yields

$$\begin{pmatrix} 1 & 3 \\ 7 & 3 \end{pmatrix}$$

The first-order Taylor approximation is then given by

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} \approx \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 7 & 3 \end{pmatrix} \begin{pmatrix} x_t - 1 \\ y_t - 1 \end{pmatrix}$$

Note that this is the linearization of the nonlinear system around the steady state.

(c) In order to find the eigenvalues, we will solve

$$\begin{vmatrix} 1-r & 3 \\ 7 & 3-r \end{vmatrix} = (1-r)(3-r) - 21 = 3 - 3r - r + r^2 - 21 = r^2 - 4r - 18 = 0$$

$$\iff r = \frac{4 \pm \sqrt{16+72}}{2} = 2 \pm \sqrt{22} \approx \begin{cases} 6.69 \\ -2.69 \end{cases}$$

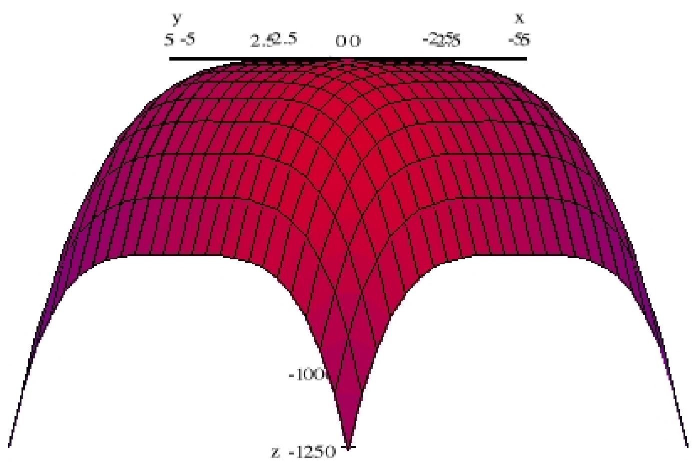
Thus, both of the eigenvalues are greater that one in absolute value, such that the steady state is locally unstable.

5. Extremal values: minima, maxima and saddle points

(a) Consider first the function

$$f(x,y) = -x^4 - y^4$$

The first-order conditions for maximum are given by



$$\frac{\partial f(x,y)}{\partial x} = -4x^3 = 0 \iff x = 0$$

$$\frac{\partial f(x,y)}{\partial y} = -4y^3 = 0 \iff y = 0$$

and the second-order conditions by

$$\frac{\partial^2 f(x,y)}{\partial x^2} = -12x^2$$

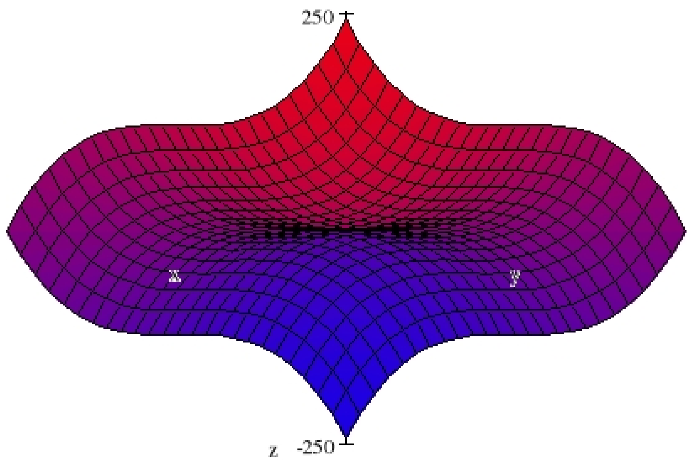
$$\frac{\partial^2 f(x,y)}{\partial y^2} = -12y^2$$

$$\frac{\partial^2 f(x,y)}{\partial xy} = \frac{\partial^2 f(x,y)}{\partial yx} = 0$$

At $(0,0)$ both derivatives obtain zero values. However, we can see that the function is concave in its domain such that it obtains its maximum at this point.

(b) Next consider

$$f(x,y) = x^3 + y^3$$



The first-order conditions

$$\frac{\partial f(x,y)}{\partial x} = 3x^2 = 0 \iff x = 0$$

$$\frac{\partial f(x,y)}{\partial y} = 3y^2 = 0 \iff y = 0$$

and the second-order conditions

$$\frac{\partial^2 f(x,y)}{\partial x^2} = 6x$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = 6y$$

$$\frac{\partial^2 f(x,y)}{\partial xy} = \frac{\partial^2 f(x,y)}{\partial yx} = 0$$

The Hessian matrix for the problem is

$$\begin{pmatrix} \frac{\partial^2 f(x,y)}{\partial x^2} & \frac{\partial^2 f(x,y)}{\partial xy} \\ \frac{\partial^2 f(x,y)}{\partial xy} & \frac{\partial^2 f(x,y)}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 6x & 0 \\ 0 & 6y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which is indefinite at $(0,0)$ such that we have a saddle point.

(c) Next

$$f(x,y,z) = x^2 + 2y^2 + 3z^2 + 2xy + 2xz$$

The first-order conditions

$$\frac{\partial f(x,y,z)}{\partial x} = 2x + 2y + 2z = 0$$

$$\frac{\partial f(x,y,z)}{\partial y} = 4y + 2x = 0 \iff 2y = x$$

$$\frac{\partial f(x,y,z)}{\partial z} = 6z + 2x = 0$$

or

$$\begin{aligned} 3x + 2z &= 0 \\ 2x + 6z &= 0 \end{aligned}$$

That is,

$$\begin{aligned} 9x + 6z &= 0 \\ 2x + 6z &= 0 \end{aligned}$$

which yields

$$7x = 0$$

and therefore

$$x = y = z = 0$$

Second-order conditions can be written in a matrix form

$$H = \begin{pmatrix} \frac{\partial^2 f(x,y,z)}{\partial x^2} & \frac{\partial^2 f(x,y,z)}{\partial xy} & \frac{\partial^2 f(x,y,z)}{\partial xz} \\ \frac{\partial^2 f(x,y,z)}{\partial xy} & \frac{\partial^2 f(x,y,z)}{\partial y^2} & \frac{\partial^2 f(x,y,z)}{\partial yz} \\ \frac{\partial^2 f(x,y,z)}{\partial xz} & \frac{\partial^2 f(x,y,z)}{\partial yz} & \frac{\partial^2 f(x,y,z)}{\partial z^2} \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 6 \end{pmatrix}$$

Now since

$$H_{11} = 2 > 0, \begin{vmatrix} H_{11} & H_{21} \\ H_{12} & H_{22} \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ 2 & 4 \end{vmatrix} = 4 > 0, \det H = \begin{vmatrix} 2 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 6 \end{vmatrix} = 8 > 0$$

the Hessian is positive definite. Thus the function has a strict minimum at $(0,0,0)$.

(d) Consider the following function

$$f(x) = \|Ax - b\|^2 + \delta \|x\|^2, \delta > 0, x \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}$$

Note that we can write

$$\begin{aligned} f(x) = \|Ax - b\|^2 + \delta \|x\|^2 &= (Ax)^T Ax - 2b^T (Ax) + b^T b + \delta \|x\|^2 \\ &= x^T A^T Ax - 2b^T (Ax) + b^T b + \delta \|x\|^2 \end{aligned}$$

The first-order conditions with respect to x are the given by

$$\begin{aligned} \nabla f(x) = \left[A^T A + \left(A^T A\right)^T\right] x - 2A^T b + 2\delta I x = 0 \\ \iff \left[A^T A + AA^T\right] x + 2\delta I x = 2A^T b \\ \iff x = 2\left[A^T A + AA^T + 2\delta I\right]^{-1} A^T b \end{aligned}$$

The second-order condition

$$\nabla^2 f(x) = A^T A + AA^T + 2\delta I$$

Note that the matrix $A^T A$ is positive semidefinite. By adding a strictly positive matrix $2\delta I$, we will assure that the solution indeed is a minimum.

6. A linear regression model: ordinary least squares

$$f(a,b) = \sum_{i=1}^{n} (a + bx_i - y_i)^2$$

(a) The problem is to

$$\min_{a,b} \sum_{i=1}^{n} (a + bx_i - y_i)^2$$

The first-order conditions

$$2\sum_{i=1}^{n} (a + bx_i - y_i) = 0 \iff \sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i$$

$$2\sum_{i=1}^{n} x_i (a + bx_i - y_i) = 0 \iff \sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2$$

Multiplying by $\sum_{i=1}^{n} x_i$ and by n yields

$$\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i = na\sum_{i=1}^{n} x_i + b\left(\sum_{i=1}^{n} x_i\right)^2$$

$$n\sum_{i=1}^{n} x_i y_i = an\sum_{i=1}^{n} x_i + bn\sum_{i=1}^{n} x_i^2$$

Subtracting yields

$$\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i - n\sum_{i=1}^{n} x_i y_i = b\left[\left(\sum_{i=1}^{n} x_i\right)^2 - n\sum_{i=1}^{n} x_i^2\right]$$

from which it follow that

$$b = \frac{\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i - n\sum_{i=1}^{n} x_i y_i}{\left(\sum_{i=1}^{n} x_i\right)^2 - n\sum_{i=1}^{n} x_i^2}$$

Given b, we may calculate a from

$$a = \frac{\sum_{i=1}^{n} y_i}{n} - b\frac{\sum_{i=1}^{n} x_i}{n}$$

Substituting for the points $(-1, 2), (0, 1)$, and $(1, 0)$ we obtain

$$\sum_{i=1}^{3} x_i = 0, \quad \sum_{i=1}^{3} y_i = 3, \quad \sum_{i=1}^{3} x_i^2 = 2, \quad \sum_{i=1}^{3} x_i y_i = -2$$

Furthermore,

$$b = \frac{0 \cdot 3 - 3 \cdot (-2)}{0^2 - 3 \cdot 2} = -1$$

and therefore,

$$a = \frac{3}{3} - 1 \cdot \frac{0}{3} = 1$$

(b) Now we add a fourth observation point, $(7, 8)$

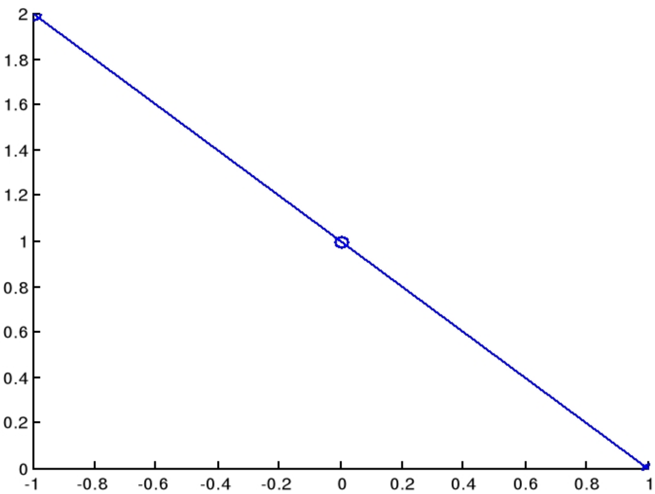
$$\sum_{i=1}^{3} x_i = 7, \quad \sum_{i=1}^{3} y_i = 11, \quad \sum_{i=1}^{3} x_i^2 = 51, \quad \sum_{i=1}^{3} x_i y_i = 54$$

The regression coefficients are then

$$b = \frac{7 \cdot 11 - 3 \cdot 54}{7^2 - 3 \cdot 51} \approx 0.817$$

and

$$a = \frac{11}{3} - 0.817 \cdot \frac{7}{3} \approx 1.760$$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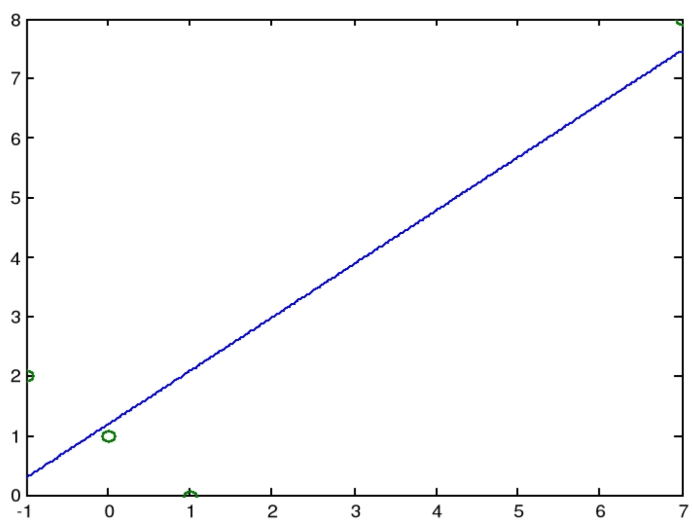
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