1 Metric Spaces

Definition 1 *A set X is said to be a metric space if with any two points p and q of X there is associated a real number $d(p,q)$, called the distance from p to q, such that:*

(i) $d(p,q) > 0$ *if* $p \neq q$; $d(p,p) = 0$,

(ii) $d(p,q) = d(q,p)$,

(iii) $d(p,q) \leq d(p,r) + d(r,q)$, *for any* $r \in X$.

Any function with these three properties is called a distance function or a metric.

Example 1 *Let X be any set. Then the function defined by $d(x,x) = 0$ for all $x \in X$, $d(x,y) = 1$ for all $x, y \in X$, $x \neq y$ is a distance. Hence any set cam be made into a metric space.*

Example 2 *Let X be the set of real valued functions defined on $[0,1]$. If $f, g \in X$, define $d(f,g)$ by*

$$d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

It is easy to verify that $d(f,g)$ is indeed a distance.

The next subsection considers the most important example of a metric space for our purposes.

1.1 Length and Distance in $\mathbb{R}^n$

The only spaces that we will be interested in are the various Cartesian products of the real line $\mathbb{R}$ denoted by $\mathbb{R}^n$. The exponent n is also called the dimension of the Euclidean space. Hence an element $x \in \mathbb{R}^n$ is an ordered

n-tuple $(x_1, ..., x_n)$ where each $x_i \in \mathbb{R}$. The Euclidean norm or the length of a vector in $\mathbb{R}^n$ is defined by

$$\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}. \tag{1}$$

A distance for $\mathbb{R}^n$ can be derived from this norm as

$$d(x, y) = \|x - y\|.$$

Proposition 1 *Let x and y denote points in $\mathbb{R}^n$. Then we have:*
(*a*) $\|x\| \geq 0$ *and* $\|x\| = 0$ *if and only if* $x = \mathbf{0}$*,*
(*b*) $\|ax\| = a\|x\|$ *for every real* a*,*
(*c*) $\|x - y\| = \|y - x\|$,
(*d*) $\|x \cdot y\| \leq \|x\| \cdot \|y\|$ (*Cauchy-Schwarz inequality*),
(*e*) $\|x + y\| \leq \|x\| + \|y\|$ (*triangle inequality*).

By the segment (a, b) we mean the set of all real number x such that $a < x < b$. By the interval $[a, b]$, we mean the set of all real numbers such that $a \leq x \leq b$. If $a_i < b_i$ for $i = 1, ..., k$, the set of all points $\mathbf{x} = (x_1, ..., x_k)$ in $\mathbb{R}^k$ whose coordinates satisfy $a_i \leq x_i \leq b_i$ $(1 \leq i \leq k)$, is called a k-cell. If $\mathbf{x} \in \mathbb{R}^n$ and $r > 0$, the open or closed ball B with center at x and radius r is defined to be the set of all $y \in \mathbb{R}^n$, such that $|y - x| < (\leq) r$. We call a set $E \in \mathbb{R}^n$ convex if

$$\lambda x + (1 - \lambda) y \in E,$$

whenever $x \in E, y \in E$, and $0 \leq \lambda \leq 1$.

2 Open and closed sets

Definition 2 *Let X be a metric space. All points and sets mentioned are understood to be elements and subsets of X.*

(a) *A neighborhood of a point* p *is a set* $N_r(p)$, *consisting of all points of* q *such that* $d(p,q) < r$. *The number* r *is called the radius of* $N_r(p)$.

(b) *A point* p *is alimit point of the set* E *if every neighborhood of* p *contains a point* $q \neq p$, *such that* $q \in E$.

(c) *If* $p \in E$ *and* p *os not a limit point of E, then* p *is called an isolated point of E.*

(d) *E is closed if every limit point (cluster point) of E is a point of E.*

(e) *A point* p *is an interior point of E if there is a neighborhood* N *of* p *such that* $N \subset E$.

(f) *E is open if every point of E is an interior point.*

(g) *The complement of E, denoted* E^c *is the set of all points* $p \in X$ *such that* $p \in X$, *such that* $p \notin E$.

(h) *E is bounded if there is a real number* M *and a point* $q \in X$ *such that* $d(p,q) < M$ *for all* $p \in E$.

(i) *E is dense in X if every point of X is a limit point of E, or a point of E,* (*or both*).

Proposition 2 *A set E is open if and only if its complement is closed. A set F is closed if and only if its complement is open.*

Proposition 3 (i) *For any collection* $\{G_\alpha\}$ *of open sets,* $\bigcup_\alpha G_\alpha$ *is open.*

(ii) *For any collection* $\{F_\alpha\}$ *of closed sets,* $\bigcap_\alpha F_\alpha$ *is closed.*

(iii) *For any finite collection* $G_1, ..., G_n$ *of open sets,* $\bigcap_{i=1}^n G_i$ *is open.*

(iv) *For any finite collection* $F_1, ..., F_n$ *of closed sets,* $\bigcup_{i=1}^n F_i$ *is closed.*

Definition 3 *If X is a metric space, if* $E \subset X$, *and if* E' *denotes the sets of all limit points of E in X, then the closure of E is the set* $\overline{E} = E \cup E'$.

3 Compact Sets

Definition 4 *By an open cover of a set E in a metric space X, we mean a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \bigcup_\alpha G_\alpha$.*

Definition 5 *A subset K of a metric space X is said to be compact if every open cover of K contains a finite subcover.*

Theorem 4 ***(Heine Borel).*** *Let F be an open covering of a closed and bounded set A in $\mathbb{R}^n$. Then a finite subcollection of F also covers A.*

Proposition 5 *Suppose $K \subset Y \subset X$. Then K is compact relative to X if and only if K is compact relative to Y.*

Proposition 6 *Compact subsets of metric spaces are closed. Closed subsets of compact spaces are compact.*

Theorem 7 *If a set E in $\mathbb{R}^k$ has one of the following three properties, then it has the other two:*
- (i) *E is closed and bounded.*
- (ii) *E is compact.*
- (iii) *Every infinite subset of E has a limit point in E.*

Theorem 8 ***(Weierstrass)*** *Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.*

Definition 6 *Two subsets A and B of a metric space X are said to be separated if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty. A set $E \subset X$ is said to be connected if E is not the union of two nonempty separated sets.*

4 Sequences

Definition 7 *If S is any set, a sequence in S is a function on the set $\mathbb{N} = \{1, 2, 3, ...\}$ of natural numbers and whose range is in S.*

Remark 1 *If $X : \mathbb{N} \rightarrow \mathbb{R}^n$ is a sequence in $\mathbb{R}^n$, the value of X at $n \in \mathbb{N}$ should be symbolized by $X(n)$ rather than x_n. The conventional symbolism, to which we shall adhere is to denote the function by $X = \{x_n\}$ or $X = \{x_n; n \in \mathbb{N}\}$.*

Definition 8 *A sequences $\{x_n\}$ in a metric space X is said to converge if there is a point $x \in X$ with the following property: For every $\epsilon > 0$, there is an integer N such that $n \geq N$ implies that $d(x_n, x) < \epsilon$.*

We say that x_n converges to x, x is the limit of $\{x_n\}$ and we write $x_n \rightarrow x$,

$$\lim_{n\rightarrow\infty} x_n = x.$$

Notice that the notion of convergence of a sequence depends crucially on how the distance is defined. Consider e.g. the sequence of functions defined on $[0, 1]$ by

$$f_n = \max\{1 - nx, 0\}.$$

Then it is clear that for all $x \in (0, 1]$, $\lim_n f_n(x) = 0$ and $\lim_n f_n(0) = 1$. Hence the f_n converge pointwise ot the function

$$f = \begin{cases} 1 \text{ if } x = 0, \\ 0 \text{ if } x > 0. \end{cases}$$

Notice however that if we define a distance on the set of all real valued functions on $[0, 1]$ by

$$d(f, g) = \sup_{x\in[0,1]} |f(x) - g(x)|,$$

then it is not true that $f_n \rightarrow f$.

Theorem 9 *Let $\{x_n\}$ be a sequence in a metric space X.*

(i) $\{x_n\}$ converges to $x \in X$ if and only if every neighborhood of x contains all but finitely many of the terms of $\{x_n\}$.

(ii) If $x \in X, x' \in X$, and if $\{x_n\}$ converges to x and to x', then $x = x'$.

(iii) If $\{x_n\}$ converges, then $\{x_n\}$ is bounded.

(iv) If $E \subset X$ and x is a limit point of E, then there is a sequence $\{x_n\}$ in E such that $x = \lim_{n\to\infty} x_n$.

Definition 9 *Given a sequence $\{x_n\}$, consider an infinite sequence $\{n_k\}$ of positive integers, such that $n_1 < n_2 < \cdots$. Then the sequence $\{x_{n_i}\}$ is called a subsequence of $\{x_n\}$. If $\{x_{n_i}\}$ converges, its limit is called a subsequential limit of $\{x_n\}$.*

Proposition 10 *(i) If a sequence $\{x_n\}$ converges in a metric space to x, then any subsequence of X also converges x.*

(ii) If $\{x_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{x_n\}$ converges to a point of X.

Theorem 11 ***(Bolzano-Weierstrass Theorem)***

Every bounded sequence in $\mathbb{R}^n$ contains a convergent subsequence.

Proposition 12 *Let X and Y be sequences in $\mathbb{R}^n$ which converge to x and y, respectively. Then the sequences $X + Y, X - Y$ and $X \cdot Y$ converge to $x + y, x - y$, and $x \cdot y$, respectively.*

Definition 10 *A sequence $\{x_n\}$ in a metric space is said to be a Cauchy sequence if for every $\epsilon > 0$ there is an integer N such that $d(x_n, x_m) < \epsilon$, if $n \geq N$ and $m \geq N$.*

Definition 11 *A sequence $\{x_n\}$ in $\mathbb{R}^n$ is monotone increasing if*

$$x_1 \leq x_2 \leq .... \leq x_n \leq x_{n+1} \leq ...$$

and monotone decreasing if

$$x_1 \geq x_2 \geq .... \geq x_n \geq x_{n+1} \geq ... .$$

Theorem 13 ***(Monotone Convergence)***

Any monotone sequence $\{x_n\}$ converges if and only if it is bounded in which case

$$\lim x_n = \sup x_n, \ \ or \quad \lim x_n = \inf x_n.$$

Proposition 14 (i) *In any metric space X, every convergent sequence is a Cauchy sequence.*

(ii) *If X is a compact metric space and if $\{x_n\}$ is a Cauchy sequence in X, then $\{x_n\}$ converges to some point of X.*

(iii) *In $\mathbb{R}^n$, every Cauchy sequence converges* ***(Cauchy Convergence Criterion)****.*

Definition 12 *A metric space in which every Cauchy sequence converges is said to be complete.*

5 Limit Superior

Definition 13 *Let $X = \{x_n\}$ be a bounded sequence in $\mathbb{R}$.*

(a) *The limit superior of X, which we denote by*

$$\limsup X, \quad \limsup \ (x_n), \quad or \quad \overline{\lim}\,(x_n),$$

is the infimum of the set V of $v \in \mathbb{R}$ such that there are at most a finite number of $n \in N$ such that $v < x_n$.

(b) *The limit inferior of X which we denote by*

$$\liminf X, \quad \liminf (x_n), \quad or\ \underline{lim}\,(x_n),$$

is the supremum of the set W of $w \in \mathbb{R}$ such that there are at most a finite number of $m \in N$, such that $x_m < w$.

Theorem 15 *If $X = (x_n)$ is a bounded sequence in $\mathbb{R}$, then the following statements are equivalent for a real number x^* :*

(a) $x^* = \limsup (x_n)$,

(b) *If $\epsilon > 0$, there are at most a finite number of $n \in N$ such that $x^* + \epsilon < x_n$, but there are an infinite number such that $x^* - \epsilon < x_n$.*

(c) *If $v_m = \sup \{x_n : n \geq m\}$, then $x^* = \inf \{v_m : n \in N\}$.*

(d) *If $v_m = \sup \{x_n : n \geq m\}$, then $x^* = \lim(v_m)$.*

(e) *If L is the set of $v \in \mathbb{R}$ such that there exists a subsequence of X which converges to v, then $x^* = \sup L$.*

6 Continuous Functions

Definition 14 *Let X and Y be metric spaces; suppose $E \subset X$, f maps E into Y, and p is a limit point of E. We write $f(x) \to q$ as $x \to p$, or*

$$\lim_{x \to p} f(x) = q, \tag{2}$$

if there is a point $q \in Y$ with the following property: For every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), q) < \epsilon$$

for all points $x \in E$ of which

$$0 < d_X(x, p) < \delta.$$

The definition can also be formulated in terms of balls. Thus (2) holds if, and only if, for every ball $B_Y(q)$ there is a ball $B_X(p)$ such that $B_X(p) \cap E$ is not empty and such that

$$f(x) \in B_Y(q)$$

whenever $x \in B_X(p) \cap A,\ x \neq p$.

Definition 15 *Suppose* X *and* Y *are metric spaces,* $E \subset X$*,* $p \in E$*, and* f *maps* E *into* Y*. Then* f *is said to be continuous at* p *if for every* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$d_Y\left(f\left(x\right), f\left(p\right)\right) < \epsilon$$

for all points $x \in E$ *for which* $d_X\left(x, p\right) < \delta$*.*

Continuity of a function f at a point p is called a local property of f because it depends on the behavior of f only in the immediate vicinity of p. A property of f which concerns the whole domain of f is called a global property. Thus, continuity of f on its domain is a global property.

Proposition 16 *A mapping* f *of a metric space* X *into a metric space* Y *is continuous if and only if* $f^{-1}\left(V\right)$ *is open* (*closed*) *in* X *for every open* (*closed*) *set* V *in* Y*.*

Proposition 17 *Let* $f : X \rightarrow Y$ *be a function from one metric space* X *to another* Y*. Then* f *is continuous at* p *if, and only if, for every sequence* $\{x_n\}$ *in* X *convergent to* p*, the sequence* $\{f\left(x_n\right)\}$ *in* Y *converges to* $f\left(p\right)$*; in symbols,*

$$\lim_{n\rightarrow\infty} f\left(x_n\right) = f\left(\lim_{n\rightarrow\infty} x_n\right).$$

Proposition 18 *Let* S*,* T*,* U *be metric spaces. Let* $f : S \rightarrow T$ *and* $g : f\left(S\right) \rightarrow U$ *be functions, and let* h *be the composite function defined on* S *by the equation*

$$h\left(x\right) = g\left(f\left(x\right)\right) \qquad \text{for } x \text{ in } S.$$

If f *is continuous at* p *and if* g *is continuous at* $f\left(p\right)$*, then* h *is continuous at* p*.*

Definition 16 ***(Topological mapping, homeomorphism)***

Let $f : X \to Y$ be a function from one metric space X to another Y. Assume also that f is one-to-one on X, so that the inverse function f^{-1} exists. If f is continuous on X and if f^{-1} is continuous on $f(S)$, then f is called a topological mapping or a homeomorphism, and the metric space X and $f(X)$ are said to be homeomorphic.

7 Global Properties of Continuous Functions

Definition 17 *A mapping f of a set E into $\mathbb{R}$ is said to be bounded if there is a real number M such that $|f(x)| \leq M$ for all $x \in E$.*

Proposition 19 *Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact.*

Proposition 20 *If f is a continuous mapping of a compact metric space X into $\mathbb{R}$ then $f(X)$ is closed and bounded, thus f is bounded.*

Proposition 21 *Suppose f is a continuous real function on a compact metric space X, and*

$$M = \sup_{p \in X} f(p), \qquad m = \inf_{p \in X} f(p).$$

Then there exists a point $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.

Proposition 22 *Suppose f is a continuous 1-1 mapping of a compact metric space X onto a metric space Y. Then the inverse mapping f^{-1} defined on Y by*

$$f^{-1}(f(x)) = x$$

is a continuous mapping of Y onto X.

Theorem 23 ***(Bolzano)***

Let f be real-valued and continuous on a compact interval $[a, b]$ in $I\!R$, and suppose that $f(a)$ and $f(b)$ have opposite signs. Then there is at least one point c in the open interval (a, b) such that $f(c) = 0$.

Theorem 24 ***(Intermediate value theorem).***

Assume f is real-valued and continuous on a compact interval S in $I\!R$. Suppose there are two points $\alpha < \beta$ such that $f(\alpha) \neq f(\beta)$. Then f takes every value between $f(\alpha)$ and $f(\beta)$ in the interval (α, β).

8 Uniform and Lipschitz continuity

Definition 18 *Let $f : S \to T$ where S and T are metric spaces. Then f is said to be uniformly continuous on a subset A of S if the following holds. For every $\epsilon > 0$ there exists a $\delta > 0$ (depending only on ϵ), such that if $x \in A$ and $p \in A$, then*

$$d_T(f(x), f(p)) < \epsilon \qquad \text{whenever} \quad d_S(x, p) < \delta. \tag{3}$$

Definition 19 *Suppose $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^n$. A function $f : X \to Y$ is Lipschitz continuous if there exists some scalar $\lambda > 0$, such that $\|f(x) - f(x')\| \leq \lambda \cdot \|x - x'\|$, $\forall x, x' \in X$. The function f is called a contraction mapping if $\lambda < 1$.*

Theorem 25 ***(Heine)***. *Let Let $f : S \to T$ where S and T are metric spaces. Let A be a compact subset of S and assume that f is continuous on A. Then f is uniformly continuous on A.*

Definition 20 *Let f be defined on (a, b). Consider any point x such that $a \leq x < b$. We write*

$$f(x_+) = q$$

if $f(t_n) \to q$ *as* $n \to \infty$, *for all sequences* $\{t_n\}$ *in* (x, b) *such that* $t_n \to x$. *To obtain the definition of* $f(x_-)$, *for* $a < x \leq b$, *we restrict ourselves to sequences* $\{t_n\}$ *in* (a, x).

Definition 21 *Let f be defined on* (a, b). *If f is discontinuous at a point* x, *and if* $f(x_+)$ *and* $f(x_-)$ *exist, then f is said to have a discontinuity of the first kind, or simple discontinuity, at x. Otherwise the discontinuity is of second kind.*

Definition 22 *Let* f *be real on* (a, b). *Then* f *is said to be monotonically increasing on* (a, b), *if* $a < x < y < b$ *implies* $f(x) \leq f(y)$. *If the last inequality is reversed, we obtain the definition of monotonically decreasing function.*

9 Sequences of Functions

Definition 23 *Suppose* $\{f_n\}$, $n = 1, 2, 3, ...$, *is a sequence of functions defined on a set E, and suppose that the sequence of numbers* $\{f_n(x)\}$ *converges for every* $x \in E$. *We can then define a function f by*

$$f(x) = \lim_{n \to \infty} f_n(x) \qquad (x \in E).$$

We say that $\{f_n\}$ converges on E and that f is the limit or the limit function of $\{f_n\}$. Often we say $\{f_n\}$ converges to f pointwise on E.

Definition 24 *We say that a sequence of functions* $\{f_n\}$, $n = 1, 2, 3, ...$, *converges uniformly on E to a function f if for every* $\epsilon > 0$ *there is an integer* N *such that* $n \geq N$ *implies*

$$|f_n(x) - f(x)| \leq \epsilon$$

for all $x \in E$.

Proposition 26 *The sequence of functions $\{f_n\}$ defined on E, converges uniformly on E if and only if for every $\epsilon > 0$ there exists an integer N such that $m \geq N,\ n \geq N,\ x \in E$ implies*

$$|f_n(x) - f_m(x)| \leq \epsilon.$$

Theorem 27 *Suppose $f_n \to f$ uniformly on a set E in a metric space. Let X be a limit point of E, and suppose that*

$$\lim_{t \to x} f_n(t) = A_n.$$

Then $\{A_n\}$ converges, and

$$\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n.$$

In other words the conclusion is that

$$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t).$$

Observe that an immediate corollary to this result is that if $\{f_n\}$ are all continuous and $f_n \to f$ uniformly, then f is a continuous function. The notion of uniform convergence is in general needed when the order of various limiting processes must be changed. These issues are particularly delicate in the treatment of integrals of a sequence of functions converging to a limit function.

10 Correspondence

The following generalization of the notion of continuity useful in some areas of analysis

Definition 25 *A function $f{:}D \to \mathbb{R}$ is said to be upper semi-continuous at a point c in D in case*

$$f(c) = \limsup_{x \to c} f.$$

Similarly, a function $f{:}D \to \mathbb{R}$ *is said to be lower semi-continuous at a point* c *in* D *in case*

$$f(c) = \liminf_{x \to c} f.$$

It is possible to show that if X is a compact subset of $\mathbb{R}^n$ and f is upper semi-continuous on X, then f is bounded above on X and there exists a point in X where f attains its supremum. Thus upper semi-continuous functions on compact sets possess some of the properties we have established for continuous functions, even though an upper semi-continuous function can have many points of discontinuity.

Definition 26 *A correspondence* φ *of a set* S *into a set* T *is a "rule" which associates to every element* $x \in S$ *a non-empty subset* $\varphi(x) \subset T$.

Given a correspondence φ of S into T. We consider the subset G_φ in $S \times T$ defined by

$$G_\varphi = \{(x, y) \in S \times T \mid y \in \varphi(x)\},$$

where G_φ is called the graph of φ.

Definition 27 *Let* $\varphi : X \to Y$ *and* $x \in X$. φ *is upper hemi-continuous (u.h.c.) at* $x \in X$ *if for every open set* B *such that* $\varphi(x) \subset B$, *there exists a neighborhood* A *of* x *such that* $\varphi(x') \subset B, \forall x' \in A \cap X$.

Thus a correspondence φ is said to be *u.h.c.* at the point $x \in S$, if the set $\varphi(x)$ does not "explode" if one changes the argument of x slightly. Note, however, that we do allow the set $\varphi(x)$ to suddenly become much smaller, thus the set $\varphi(x)$ may "implode". Obviously, a function f of S into T is *u.h.c.* if and only if it is continuous. This fact explains the somewhat esoteric term "hemi". Indeed a function of S into $\mathbb{R}$ may be upper semi-continuous without being continuous.

Definition 28 *φ is lower hemi-continuous (l.h.c.) at $x \in X$ if for every open set B such that $\varphi(x) \cap B \neq \emptyset$, there exists a neighborhood A of x such that $\varphi(x') \cap B \neq \emptyset, \forall x' \in A \cap X$.*

A correspondence φ of S into T is said to be *l.h.c.* if the set $\varphi(x)$ does not "suddenly become much smaller with a small change in point x". In the other words $\varphi(x)$ does not "implode"if one moves the argument x slightly. However the set $\varphi(x)$ may "explode". As in the case of *u.h.c.* a function is continuous if and only if it *l.h.c.* A correspondence $\varphi : X \to Y$ is hence said to be continuous at $x \in X$ if it is both upper hemi-continuous and lower hemi-continuous. Thus the two concepts of continuity, *u.h.c.* and *l.h.c.*, which are quite different for general correspondences, coincide and are equivalent to continuity in the case of functions. It is therefore multi-valuedness which plays a crucial role in making these concepts distinct from each other.

Theorem 28 *The compact-valued correspondence φ of S into T is u.h.c. at x if and only if for every sequence $\{x_n\}$ converging to x in S and every sequence $\{y_n\}$ with $y_n \in \varphi(x_n)$ there exists a converging subsequence of $\{y_n\}$ whose limit belongs to $\varphi(x)$.*

Theorem 29 *The correspondence φ of S into T is l.h.c. at x if and only if for every sequence $\{x_n\}$ converging to x and every $y \in \varphi(x)$, there exists a sequence $\{y_n\}$ converging to y with $y_n \in \varphi(x_n)$.*

Definition 29 *A correspondence φ of S into T is said to be closed if the graph*
$G_\varphi = \{(x, y) \in S \times T \mid y \in \varphi(x)\}$ of φ is a closed subset in $S \times T$.

Theorem 30 *Suppose T is compact and the correspondence $\varphi : S \to T$ closed-valued. Then φ is u.h.c. iff graph (φ) is closed.*

Any continuous function $f : X \to Y$ such that $f(x) \in \varphi(x)$, $x \in X$ is called a continuous selection from φ.

Lemma 31 ***(Michael's Selection Lemma)*** *Every compact-valued, convex-valued and lower hemi-continuous correspondence has a continuous selection.*

We finally state Berge's Maximum Theorem. Let $X \subset \mathbb{R}^n$, let $Y \subset \mathbb{R}^n$, let $f : X \times Y \to \mathbb{R}$ be a (single-valued) function, and let $\Gamma : X \to Y$. be a (non-empty, possibly multi-valued) correspondence. Our interest is in problems of the form

$$h(x) = \max_{y \in \Gamma(x)} f(x, y) \tag{4}$$

with

$$G(x) = \{y \in \Gamma(x) : f(x, y) = h(x)\},$$

which we call the solution correspondence. We would like to know under what conditions do the function $h(x)$ and the correspondence $G(x)$ defined by the maximization problem in (**??**) vary continuously with x.

Theorem 32 ***(Berge's Maximum Theorem).*** *Suppose the evaluation function $f : X \times Y \to \mathbb{R}$ is continuous and the constraint correspondence $\Gamma : X \to Y$ compact-valued and continuous. Then the solution correspondence $G : X \to Y$ is upper hemi-continuous and compact-valued and the value function $h : S \to \mathbb{R}$ is continuous.*

11 Fixed Point Theorems

If f is a function with domain $D(f)$ and range in the same space $\mathbb{R}^n$, then a point u in $D(f)$ is said to be a fixed point in case $f(u) = u$. A number of important results can be proved on the basis of the existence of fixed points of functions so it is of importance to have some affirmative criteria in this direction.

Theorem 33 ***(Contraction Mapping theorem).***

If $X \subset \mathbb{R}^n$ is nonempty and closed and $f : X \to X$ is a contraction mapping, then f has exactly one fixed point.

The sequence $\{x_t\}$ generated by the iterative procedure $x_{t+1} = f(x_t)$ converges to this fixed point from any initial point $x_0 \in X$. The contraction theorem established has certain advantages: it is instructive and it guarantees an unique fixed-point. However it has the disadvantage that the requirement that f be a contraction is a very severe restriction.

Theorem 34 ***(Brouwer's Fixed Point Theorem).*** *If $X \subset \mathbb{R}^n$ is nonempty, compact and convex, and $f : X \to X$ is continuous then f has at least one fixed point.*

Theorem 35 ***(Kakutani's Fixed Point Theorem).*** *If $X \subset \mathbb{R}^n$ is nonempty, compact and convex, and $\varphi : X \to X$ is convex-valued, closed-valued and upper hemi-continuous then φ has at least one fixed point.*