

1 Logic

1.1 Propositions

Definition 1 *A proposition is a statement which can be declared to be true or false without ambiguity.*

We say that “true” and “false” are the two possible truth values of a proposition.

Example 1 “ $2 + 2 = 4$ ”, “*Graduate students in Economics do not need to understand mathematics*”, “ $x < 2$ ” is not a proposition since we cannot declare the truth value without knowing the value of x . It is a propositional function.

Definition 2 *A propositional or (sentential) function is a statement which becomes a proposition when we replace the variable with a specified value.*

We denote propositions with capital letters, like A , and propositional functions indicating the variable, $A(x)$.

Definition 3 *A connective is a symbol which allows us to construct a new proposition \mathcal{A} from simple propositions A, B, C, \dots , where \mathcal{A} is called a compound proposition.*

A truth table shows the possible truth values of a compound proposition for all possible truth values of simple proposition. The basic connectives are Negation (\sim or \neg),

A	\neg A
T	F
F	T

Conjunction (“and”) (\wedge),

A	B	A \wedge B
T	T	T
T	F	F
F	T	F
F	F	F

and Disjunction (“inclusive or”) (\vee),

A	B	A \vee B
T	T	T
T	F	T
F	T	T
F	F	F

Note that $A \vee B$ is T if A or B or both are true. Alternatively we can define the “exclusive or” where the compound proposition is true if either A or B is true, but not both. The Implication (“if...then...”) (\Rightarrow) is represented by the truth table

A	B	A \Rightarrow B
T	T	T
T	F	F
F	T	T
F	F	T

In $A \Rightarrow B$, A is called the “hypothesis” and B the “conclusion”. We say that “ A is a sufficient condition for B ” and “ B is a necessary condition for A ”. The connective Equivalence (“if and only if”) (\Leftrightarrow) is represented by the

truth table,

A	B	A \Leftrightarrow B
T	T	T
T	F	F
F	T	F
F	F	T

An equivalence is hence true if both propositions have the same truth value.

Definition 4 *A compound proposition is a tautology if it is always true and a contradiction if it is always false.*

Show that “ $A \vee (\neg A)$ ”, “ $A \Rightarrow A$ ”, “ $A \Rightarrow (A \vee B)$ ” are tautologies, while “ $A \wedge (\neg A)$ ” is a contradiction.

Definition 5 *Two compound propositions \mathcal{A} and \mathcal{B} are logically equivalent if $\mathcal{A} \Leftrightarrow \mathcal{B}$ is a tautology.*

Two logical equivalents will be used commonly: “ $A \Rightarrow B$ ” and “ $\neg B \Rightarrow \neg A$ ” (contraposition) as well as “ $A \Leftrightarrow B$ ” and “ $A \Rightarrow B \wedge B \Rightarrow A$ ” (double implication). Note that starting from an implication “ $A \Rightarrow B$ ” we can construct the contrapositive: “ $\neg B \Rightarrow \neg A$ ” which is equivalent to the original implication and the converse “ $B \Rightarrow A$ ”, whose truth value does not depend on the original implication.

1.2 Quantifiers

The universal quantifier (“for all”) is denoted by \forall . By $\forall x, A(x)$ we mean that $A(x)$ is true for all x . The existential quantifier (“there exists” or “for some”) is denoted by \exists . By $\exists x, A(x)$ we mean that there exists an x for which $A(x)$ is true. The numerical quantifier “there exists a unique” is denoted by $\exists!$ and $\exists! x, A(x)$ claims that there exists a unique x for which $A(x)$ is true. Finally $\neg \exists$ means “there exists no”.

Example 2 “ $\forall x \in \mathbb{R}, x > 2$ ”, $\exists x \in \mathbb{R}, x > 2$.

Negating quantified propositions works as follows. If the original proposition is $\forall x, A(x)$, its negation is $\neg(\forall x, A(x))$ which is equivalent to $\exists x, \neg A(x)$. Special care should be taken when dealing with propositions that contain multiple quantifiers. In particular, the proposition $\forall x \exists y, A(x, y)$ meaning “for all x there exists a y such that $A(x, y)$ is true” is different from $\exists x \forall y, A(x, y)$ which means that “there exists an x such that for all y $A(x, y)$ is true”.

1.3 Methods of Proof

A proof consists of a sequence of propositions one implied by the other,

$$\underbrace{H}_{\text{Hypothesis}} \Rightarrow \underbrace{A_1 \Rightarrow \cdots \Rightarrow A_n}_{\text{Intermediate steps}} \Rightarrow \underbrace{C}_{\text{Conclusion}}$$

1. **Direct Proof.** Show that $H \Rightarrow C$.
2. **Proof by contrapositive.** We know that $H \Rightarrow C$ is logically equivalent to $(\neg C) \Rightarrow (\neg H)$. So assuming that $(\neg C)$ is true we show that $(\neg H)$ is true.
3. **Proof by contradiction.** Assume $\neg(H \Rightarrow C)$. Then show that $[\neg(H \Rightarrow C) \wedge H]$ is a contradiction. Actually it is enough to show that $[(\neg C) \wedge H]$ is a contradiction because $[\neg(H \Rightarrow C) \wedge H]$ and $[(\neg C) \wedge H]$ are logically equivalent.
4. **Proof by construction.** For example, if the conclusion is in the form “ $\exists x, A(x)$ ” find a value x such that $A(x)$ is true. Note also that proof by construction is useful in disproving proposition i.e. in finding counterexamples to candidate propositions.

5. **Proof by decomposition.** $H \Rightarrow [(A_1 \vee A_2) \wedge \neg(A_1 \vee A_2)]$, then show that $A_1 \Rightarrow C$ and $A_2 \Rightarrow C$.

Examples for the different proof techniques are given below:

1. Direct. $H : x > 0, C : 2x > 0$.
 $x > 0 \Rightarrow 2x > 0$ (multiply by 2).
2. Contrapositive. $H : x > 0, c : 2x > 0$
 $2x \leq 0 \Rightarrow x \leq 0$ (multiply by $\frac{1}{2}$)
3. Contradiction. $H : x > 0, c : 2x > 0$
 $2x \leq 0 \wedge x > 0 \Rightarrow x \leq 0 \wedge x > 0$ and this is false.
4. Construction. $H : x \in \mathbb{R}, C : \exists x, x > 2$
For $x = 10$, we have $X > 2, x \in \mathbb{R}$.
5. Decomposition. $H : x \in \mathbb{R}, C : |x| \geq 0$;
Either $x \geq 0$, or $x < 0$. and then show that $x \geq 0 \Rightarrow |x| \geq 0$ and $x < 0 \Rightarrow |x| \geq 0$.

A famous example of proof by contradiction is the following.

Example 3 Hypothesis: $x^2 = 2$. Conclusion: x is not a rational number.

Proof. Suppose that the hypothesis is true and the conclusion is not true. In other words, $x^2 = 2$ and $x = \frac{m'}{n'}$ for some nonzero integers m', n' . We need to find an equivalent for of this negation of the conclusion. By writing m' and n' as products of prime numbers and cancelling common terms in the numerator and denominator, we can write $x = \frac{m}{n}$ for nonzero integers m and n such that they have no common factors. Hence we want to show that m and n having no common factors and $(\frac{m}{n})^2 = 2$ is a contradiction.

By hypothesis:

$$x^2 = \left(\frac{m}{n}\right)^2 = \frac{m^2}{n^2} = 2.$$

Multiply both sides of the last equation by n^2 (recall that n is nonzero) to obtain

$$m^2 = 2n^2.$$

Hence m^2 is an even number. As a number is even if and only if its factorization into prime numbers contains 2, this implies that m itself must be even. But then there is a p such that

$$m = 2p.$$

Thus

$$m^2 = 2n^2 \Leftrightarrow (2p)^2 = 2n^2 \Leftrightarrow 4p^2 = 2n^2 \Leftrightarrow 2p^2 = n^2.$$

But repeating the previous steps in the argument, this leads to the conclusion that there is a q such that

$$n = 2q.$$

But then m and n have a common factor and we have reached a contradiction and the original claim is thus proved. ■

1.4 Exercises

1. Construct the truth tables for the following propositions. Which are tautologies and which are contradictions?
 - (a) $[A \wedge B] \Rightarrow [(A \vee B) \Leftrightarrow B]$,
 - (b) $[(A \Rightarrow B) \wedge (B \Rightarrow C)] \Rightarrow (A \Rightarrow C)$,
 - (c) $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$,
 - (d) $(A \Rightarrow B) \Leftrightarrow (B \Rightarrow A)$.

2. Argue that $\forall x \forall y, A(x, y) \Leftrightarrow \forall y \forall x A(x, y)$ and $\exists x \exists y, A(x, y) \Leftrightarrow \exists y \exists x, A(x, y)$ and find a statement $A(x, y)$ such that $\forall x \exists y, A(x, y)$ is true but $\exists x \forall y, A(x, y)$ is false.
3. Find the negation of the following propositions: (all the variables here are taken to be real numbers).
 - a) $\forall x \geq 1, x^3 \geq x^2$.
 - b) $\forall x \geq 0 \forall a \geq 0, ax \geq 0$.
 - c) $\forall x \exists y, |x - y| < \varepsilon \Rightarrow |x^2 - y^2| < \varepsilon^2$.
 - d) (harder) $\forall \varepsilon > 0 \exists \delta > 0 \forall x \forall y, |x - y| < \delta \Rightarrow |x^2 - y^2| < \varepsilon$.
4. An irrational number x is called tricky if there is an irrational number y such that x^y is rational. Give a proof by decomposition to show that tricky numbers exist. (Hint: Consider $\sqrt{2}^{\sqrt{2}}$.)

2 Sets

A set is a collection of objects which we call its elements. When an object x is an element of a set X , one writes $x \in X$. Otherwise, one writes $x \notin X$. A set is called a *singleton* set if it contains exactly one element. The set containing no object is called *the empty set*, and is denoted ϕ . The set to which all the objects belong we wish to consider is called the *universal set* Ω .

If A and B are sets, then one writes $A = B$ if they contain exactly the same objects, i.e., if $x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \in A$. Otherwise, one writes $A \neq B$. If only the first inclusion holds, A is called a *subset* of B and one writes $A \subset B$. In particular, every set is a subset of itself (i.e., equality is permitted). If A is a subset of B and $A \neq B$, then A is called a *proper subset* of B .

Note that the empty set ϕ is a subset of *every* set A (including itself). A set is called *finite* if it has finitely many elements, otherwise it is called infinite. In particular, ϕ is finite. For any finite set A , $\#A$ will denote its number of elements. We will sometimes write $\#A = +\infty$ if a set A has infinitely many elements. Some useful (infinite) number sets:

$$\mathbb{R} = \{x : x \text{ is a real number}\},$$

$$\mathbb{R}_+ = \{x : x \text{ is a nonnegative real number}\} = \{x \in \mathbb{R} : x \geq 0\},$$

$$\mathbb{R}_{++} = \{x : x \text{ is a positive real number}\} = \{x \in \mathbb{R} : x > 0\},$$

$$\mathbf{Z} = \{x : x \text{ is an integer}\} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\},$$

$$\mathbf{N} = \{x : x \text{ is a natural number}\} = \{x : x \text{ is a positive integer}\} = \{x \in \mathbf{Z} : x > 0\},$$

$$\mathbf{Q} = \{x : x \text{ is a rational number}\} = \{x : x = m/n \text{ for some } m \in \mathbf{Z} \text{ and } n \in \mathbf{N}\}.$$

Definition 6 If A and B are sets, then the complement of B relative to A is the set of all elements of A which do not belong to B . We denote this set by $A \setminus B = \{x \in A : x \notin B\}$, related notations are $A - B$, or $A \sim B$.

Definition 7 If A is a set, then the set of all subsets is called the power set of A , denoted by $\mathcal{P}(A)$ or 2^A .

It is called the “power set” since if A has n elements, $\mathcal{P}(A)$ has 2^n elements.

2.1 Set Operations

For any pair of real numbers a and b such that $a < b$, we write $[a, b]$ and (a, b) for the two intervals $\{x \in \mathbb{R} : a \leq x \leq b\}$ and $\{y \in \mathbb{R} : a < y < b\}$, respectively. Such intervals are called *bounded* while, for example, the

interval $\mathbb{R}_+ = [0, +\infty)$ is called *unbounded*. Intervals of the first type, i.e. those containing both end points, are called *closed* while intervals of the second type, i.e., those which contain none of their two end-points, are called *open*. Intervals of the "intermediate" type $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ are neither open nor closed.

While inclusion is a (partial binary) *relation* between sets, intersection and union formation is an *operation* on sets, producing new sets from any given collection of sets. First, suppose A and B are sets. Then

$$A \cap B = \{x : x \in A \text{ and } x \in B\} \text{ and}$$

$$A \cup B = \{x : x \in A \text{ or } x \in B \text{ or both.}\}.$$

Note in particular, that for any set A , $A \cap A = A \cup A = A$, $A \cap \phi = \phi$ and $A \cup \phi = A$. Two sets are said to be *disjoint* if their intersection is (the) empty (set).

These familiar operations can be generalized from pairs of sets to arbitrarily large collections of sets. Let I be an arbitrary nonempty set (not necessarily a number set), to be called an *index* set, and for each $i \in I$, let A_i be a set. Then

$$\bigcap_{i \in I} A_i = \{x : x \in A_i \text{ for all } i \in I\} \text{ and } \bigcup_{i \in I} A_i = \{y : y \in A_i \text{ for some } i \in I\}.$$

Example 4

$$\bigcap_{n \in \mathbb{N}} [n-1, n] = \phi, \bigcup_{n \in \mathbb{N}} [n-1, n] = \mathbb{R}_+ \text{ and } \bigcup_{n \in \mathbb{N}} \left[\frac{1}{n+1}, \frac{n}{n+1} \right] = (0, 1).$$

The difference of set A and B in A is defined as follows:

$$A \setminus B = \{x \in A : x \notin B\}.$$

The complement of a set A in the universal set Ω is given by

$$A^C = \Omega \setminus A = \{x \in \Omega : x \notin A\}.$$

Definition 8 *If A and B are two non-void sets, then the Cartesian product $A \times B$ of A and B is the set of all ordered pairs (a, b) with $a \in A$, and $b \in B$.*

The Cartesian product can be extended to a finite or infinite collection of sets

$$\prod_{i=1}^N A_i = \{(x_1, \dots, x_n), x_i \in A_i, i = 1, \dots, n\}.$$

2.2 Exercises

1. Show that the set D of all elements that belong either to A or to B but not to both is given by

$$D = (A \setminus B) \cup (B \setminus A).$$

The set D is often called the **symmetric difference** of A and B and denoted by $A \triangle B$. Represent it by a Venn diagram. Show that the symmetric difference D is also given by

$$D = (A \cup B) \setminus (A \cap B).$$

2. Prove the following

a) $(A \cup B)^C = A^C \cap B^C$.

b) $(A \cap B)^C = A^C \cup B^C$.

c) $A \subset B \Leftrightarrow B^C \subset A^C$.

3 Binary Relations

Definition 9 *A binary relation from A to B is a subset $A \times B$.*

For a binary relation $\mathcal{R} \subseteq A \times B$ we will write $x\mathcal{R}y$ if and only if $(x, y) \in \mathcal{R}$ and we will say that “ x is related to y through \mathcal{R} ”.

Definition 10 Consider a relation \mathcal{R} in $A \times A$. We define the following properties. \mathcal{R} is

1. reflexive, if $\forall x \in A, x\mathcal{R}x$,
2. transitive, if $\forall x, y, z \in A, x\mathcal{R}y \wedge y\mathcal{R}z \Rightarrow x\mathcal{R}z$.
3. symmetric, if $\forall x, y \in A, x\mathcal{R}y \Rightarrow y\mathcal{R}x$.
4. antisymmetric, if $\forall x, y \in A, x\mathcal{R}y \wedge y\mathcal{R}x \Rightarrow x = y$.
5. acyclic, if $\forall x_1, \dots, x_n \in A, x_1\mathcal{R}x_2 \wedge x_2\mathcal{R}x_3 \wedge \dots \wedge x_{n-1}\mathcal{R}x_n \Rightarrow \neg x_n\mathcal{R}x_1$.
6. asymmetric, if $\forall x, y \in A, x \neq y, x\mathcal{R}y \Rightarrow \neg[y\mathcal{R}x]$.

Definition 11 A binary relation which is reflexive and transitive is called a preorder.

Definition 12 A binary relation which is reflexive, transitive and symmetric is called an equivalence relation.

Definition 13 A binary relation which is reflexive, transitive, and antisymmetric is called an order relation or simply an order.

Example 5 “Equality” in \mathbb{R} is an equivalence relation.

3.1 Orders

Given an order \mathcal{R} in A two points $x, y \in A$ are comparable if $x\mathcal{R}y$ or $y\mathcal{R}x$. If all the elements of A are comparable through \mathcal{R} , then \mathcal{R} is called a total order in A . Otherwise it is a partial order.

Example 6 The weak inequality “ \leq ” is a total order in \mathbb{R} .

Definition 14 An ordered set is a set in which an order is defined.

A set can be totally or partially ordered.

Definition 15 Suppose S is an ordered set with the order “ \leq ”, and $E \subset S$. If there exists a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that E is bounded above, and call β an upper bound of E . If there exists an $\alpha \in S$, such that $\alpha \leq x$ for every $x \in E$, we say that E is bounded below, and call α a lower bound of E .

Definition 16 Suppose S is an ordered set, $E \subset S$, and E is bounded above. Suppose there exists an $\beta \in S$ with the following properties:

- (i) β is an upper bound of E ,
- (ii) if $\gamma < \beta$, then γ is not an upper bound of E .

Then β is called the least upper bound of E or the supremum of E , and we write

$$\beta = \sup E.$$

The greatest lower bound, or infimum, is defined as

$$\alpha = \inf E,$$

which means that α is a lower bound of E and that no δ with $\delta > \alpha$ is a lower bound of E .

Definition 17 An ordered set S is said to have the least-upper-bound property if the following is true: If $E \subset S$, E is not empty, and E is bounded above, then $\sup E$ exists in S .

3.2 Functions

Definition 18 Consider two sets A and B . Suppose that with each element of $x \in A$ there is associated an element of B , which we denote by $f(x)$. Then f is said to be a function or a mapping from A to B and we write

$$f : A \rightarrow B.$$

The set A is called the domain of and the elements $f(x)$ are called the values of f . The set of all values of f is called the range of f . A function is hence a binary relation with the property that to each element of A corresponds one and only one element of B . Observe that a function is a special case of a binary relation on $A \times B$. The relation is then given by $a\mathcal{R}b$ if and only if $b = f(a)$.

Definition 19 *Let A and B be two sets and let f be a mapping of A into B . If $E \subset A$, $f(E)$ is defined to be the set of all elements $f(x)$ for $x \in E$. We call $f(E)$ the image of E under f and $f(A)$ is the range of f .*

Definition 20 *If $f(A) = B$, then f maps A onto or f is surjective.*

Definition 21 *If $E \subset B$, $f^{-1}(E)$ denotes the set of all $x \in A$ such that $f(x) \in E$. We call $f^{-1}(E)$ the inverse image of E under f^{-1} . If $y \in B$, $f^{-1}(y)$ is the set of all $x \in A$, such that $f(x) = y$.*

Definition 22 *If, for each $y \in B$, $f^{-1}(y)$ consists of at most one element of A , then f is said to be a one-to-one or injective mapping of A into B .*

Note that we could also say that f is one-to-one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Definition 23 *If there exists a 1-1 mapping f of A onto B , we say that A and B can be put in 1-1 correspondence, and f is said to be bijective.*

Observe that if a function f is bijective, then the binary relation \mathcal{R} on $B \times A$ defined by $b\mathcal{R}a$ if and only if $f^{-1}(b) = a$ is also a function, i.e. f^{-1} associates to each element in B a unique element in A .

Suppose next that we have two functions, $f : A \rightarrow B$ and $g : B \rightarrow C$. We can define a composite function $h : A \rightarrow C$ denoted by $g \circ f$ as follows.

Definition 24 Let $f : A \rightarrow B$ and $g : B \rightarrow C$. Then the composite function $h = g \circ f : A \rightarrow C$ is defined by $h(a) = g(f(a))$.

A composite function h is bijective if and only if both f and g are bijective. Notice also that the composite function of $f : A \rightarrow B$ and $f^{-1} : B \rightarrow A$ is given by the identity function $i : A \rightarrow A$ that satisfied $i(a) = a$ for all $a \in A$.

3.3 Exercises

1. Let $f : S \rightarrow T$ be a function. If $Y \subseteq T$, we denote by $f^{-1}(Y)$ the largest subsets of S which f maps into Y . That is

$$f^{-1}(Y) = \{x : x \in S \text{ and } f(x) \in Y\}.$$

The set $f^{-1}(Y)$ is called the inverse image of Y under f . Prove the following for arbitrary subsets X of S and Y of T .

- (i) $X \subseteq f^{-1}[f(X)]$,
 - (ii) $f[f^{-1}(Y)] \subseteq Y$,
 - (iii) $f^{-1}[Y_1 \cup Y_2] = f^{-1}(Y_1) \cup f^{-1}(Y_2)$,
 - (iv) $f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2)$.
2. Let $f : S \rightarrow T$ be a function. Prove that the following statements are equivalent.
 - (a) f is one-to-one on S ,
 - (b) $f(A \cap B) = f(A) \cap f(B)$ for all subsets A, B of S ,
 - (c) $f^{-1}[f(A)] = A$ for every subset A of S .
 - (d) For all disjoint subsets A and B of S , the images $f(A)$ and $f(B)$ are disjoint.

(e) For all subsets A and B of S with $B \subseteq A$, we have

$$f(A - B) = f(A) - f(B).$$

(Hint: We may define

$$f(\emptyset) = \emptyset.)$$

4 The Structure of \mathbf{N} , \mathbf{Z} , \mathbf{Q} , \mathbb{R}

4.1 Countability

Definition 25 *Two sets are called similar or equinumerous if there exists a bijective function $f : A \rightarrow B$.*

We write $A \sim B$ and say that f establishes a one-to-one correspondence between A and B .

Definition 26 *For any positive integer n , let J_n be the set whose elements are the integers $1, 2, \dots, n$; let J be the set consisting of all positive integers. For any set A we say:*

- (a) *A is finite if $A \sim J_n$ for some n ,*
- (b) *A is infinite if A is not finite,*
- (c) *A is countable if $A \sim J$,*
- (d) *A is uncountable if A is neither finite nor countable.*

The integer n is called the cardinal number of A .

Theorem 1 *Every subset of a countable set is countable.*

Example 7 *Let $\mathbf{Z} = \{0, 1, -1, 2, -2, \dots\}$ the set of integers. Then $\mathbf{N} \sim \mathbf{Z}$. Finding a 1-1 correspondence between \mathbf{N} and \mathbf{Z} is left as an easy exercise.*

Theorem 2 *If F is a countable collection of countable sets, then the union of all sets in F is also a countable set.*

Example 8 *Consider \mathbf{Q} . Denote by A_n the set of all positive rational numbers with denominator $n \in \mathbf{N} \setminus \{0\}$. The set of all positive rational numbers is equal to $\bigcup_{i=1}^{\infty} A_i$. By the previous theorem then, the set of rational numbers is countable.*

4.2 Mathematical Induction

Definition 27 *A set S of numbers is said to be inductive if and only if*

- (a) $1 \in S$,
- (b) $(x + 1) \in S$ whenever $x \in S$.

Definition 28 *A real number n is said to be a natural number if and only if it belongs to every inductive set of real numbers. The set of all natural numbers will be denoted by \mathbf{N} .*

Theorem 3 *The set \mathbf{N} of all natural numbers is an inductive set.*

Theorem 4 *Principle of mathematical induction. If S is an inductive set of natural numbers, then $S = \mathbf{N}$.*

Example. Show that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

for every natural number n . Solution. Let S be the set of natural numbers n for which the formula holds. we shall show that S is an inductive set. Clearly $1 \in S$. Suppose $k \in S$ and the formula holds with $k = n$. Adding $(k + 1)$ to both sides we see that

$$1 + 2 + \cdots + k + (k + 1) = \frac{k(k+1)}{2} + (k + 1) = \frac{(k+1)(k+2)}{2},$$

which is the formula for $n = k + 1$, which concludes that $(k + 1) \in S$.

4.3 Exercises

1. Show by induction that $n! > 2^n$ for each natural number $n > 3$. (Hint: The inductive set has two parts.)
2. For $n \in \mathbf{N}$, we define $n! = 1 \cdot 2 \cdot \dots \cdot n$. We also define $0! = 1$. Then

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

where $n \geq k$, $n \in \mathbf{N}$, $k \in \mathbf{N}$, and $\binom{n}{k}$ represents the number of different combinations of n objects in non-ordered k -tuples. Prove the binomial theorem

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k.$$

Hint: Prove first that

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}.$$