1. EXERCISE

1.
a) Show that an intersection of closed sets is closed. (Hint: you do not need metrics, think about complements and open sets.)

b) Show that $S = \{x \in \mathbb{R}^n : g(x) \leq 0\}$ is a closed set when $g : \mathbb{R}^n \mapsto \mathbb{R}$ is a continuous function. Use the definition of continuity based on convergent sequences $\{x^k\}_k$ (and the usual metric in $\mathbb{R}^n$).

c) Show that the budget set $\{x \in \mathbb{R}^n : x \geq \mathbf{0},\ p \cdot x \leq I\}$ is a compact set for $p \gg \mathbf{0}$, $I > 0$. (For the closedness you can use the results (a) and (b).)

2.
a) Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Is this function injective or surjective? Can you find a restricted domain such that f is bijective on that domain?

b) For what values of v is the linear function given by the following matrix injective? When is it surjective?

$$\begin{pmatrix} 1 & 0 & 4 \\ v & 3 & -6 \end{pmatrix}$$

3.
a) Let V be the space of polynomials on $\mathbb{R}$ with sum and scalar multiplication defined pointwise; $p + q$ and αp mean that $(p + q)(x) = p(x) + q(x)$ and $(\alpha p)(x) = \alpha p(x)$ for all $x \in \mathbb{R}$. Argue that differentiation D, $Dp = p'$, is a linear mapping on V (tip: you need not use the definition of differential). What is the kernel of D?

b) Let X be the space of all sequences of real numbers, i.e. X contains all infinite vectors of the type: $x = (x_0, x_1, x_2, ...)$. Show that X is a vector space under coordinatewise addition and scalar multiplication (just as the finite dimensional spaces are). Next define a function L on X by $L(x) = (x_1, x_2, x_3, ...)$, i.e. L shifts all the coordinates of x by one position to the left (and discards the first coordinate). Show that L is a linear function on X.

4.
a) Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear maps as indicated. Decide if L is invertible in each case. Extra: try Matlab function rank to the matrices presenting the linear maps.

(i) $L(x, y, z) = (x - y, x + z, x + y + 2z)$,

(ii) $L(x, y, z) = (2x - y + z, x + y, 3x + y + z)$.

b) Solve the following systems of linear equations either by Cramer's Rule or Gaussian Elimination. Extra try Matlab in solving the equations.

(i) $3x + y - z = 0,\ x + y + z = 0,\ y - z = 1,$

(ii) $2x - y + z = 1,\ x + 3y - 2z = 0,\ 4x - 3y + z = 2.$

5.

a) For what values of a does the following system of equations has a solution. Is the solution unique?

$$\begin{bmatrix} 3 & 4 & 1 \\ 6 & 6 & 0 \\ 6 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ a \end{bmatrix}$$

b) How many linearly independent solutions does the following system of equations have?

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 6 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

6.

Markov process determined by the following Markov matrix has been estimated for business cycles (Hamilton 2005). There are three states: normal growth (state 1), mild recession (state 2), and recession (state 3). The Markov matrix is

$$M = \begin{pmatrix} 0.97 & 0.15 & 0 \\ 0.03 & 0.77 & 0.5 \\ 0 & 0.08 & 0.5 \end{pmatrix}.$$

Find the eigenvalues of the Markov matrix and give the general solution of the difference equation $z^{k+1} = Mz^k$ by using the eigenvalues and eigenvectors. One of the eigenvalues is 1. What is the interpretation of the eigenvector corresponding to this eigenvalue when the eigenvector is normalized such that its components sum up to one, what happens when $k \to \infty$? Extra: try Matlab function eig for finding the eigenvalues and eigenvectors of M.