2. EXERCISE

1.

a) Are the following functions continuous? Are they differentiable?

(i) $f(x) = \mid x \mid$.

(ii) $f(x) = \begin{cases} 0 & \text{if } x < 0, \\ x^2 & \text{if } x \geq 0. \end{cases}$

b) (Euler's Theorem) A function $f(\mathbf{x}) = f(x_1, ..., x_n)$ is homogeneous of degree r if $f(\lambda \mathbf{x}) = \lambda^r f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$, where $\lambda \in \mathbb{R}$. Show that $\sum_{i=1}^n x_i \frac{\partial f(\mathbf{x})}{\partial x_i} = rf(\mathbf{x})$. (Hint: Differentiate both sides of the definition and set $\lambda = 1$).

2.

a) In each of the next two problems a function F and a point P are given. Verify that the implicit function theorem is applicable. Denoting the implicitly defined function by f, find the values of all the first partial derivatives of f at P.

(i) $F = (F^1, F^2)$, $P = (0,0,0,0)$, where $F^1 = 2x_1 - 3x_2 + y_1 - y_2$, $F^2 = x_1 + 2x_2 + y_1 - 2y_2$,

(ii) $F = (F^1, F^2)$, $P = (0,0,0,0)$, where $F^1 = 2x_1 - x_2 + 2y_1 - y_2$, $F^2 = 3x_1 + 2x_2 + y_1 + 2y_2$.

b) Suppose that the equilibrium in a model with two endogenous variables, y_1, y_2 and exogenous variables, α and β is given by equations:

$$f_1(y_1,\ y_2) = 3y_1y_2^2 - 2\alpha y_1 + \beta = 0$$
$$f_2(y_1,\ y_2) = 3y_1y_2 - 3y_2 + \alpha = 0.$$

Can you solve for the endogenous variables as functions of the exogenous variables in a neighborhood of point $(y_1,\ y_2, \alpha, \beta) = (1, 1, 0, -3)$? Find the matrix of partial derivatives:

$$\begin{pmatrix} \frac{\partial y_1}{\partial \alpha} & \frac{\partial y_1}{\partial \beta} \\ \frac{\partial y_2}{\partial \alpha} & \frac{\partial y_2}{\partial \beta} \end{pmatrix}.$$

3.

a) Show that the log-linearization of XY around (X^*, Y^*) is $X^*Y^*(1+x-x^*+y-y^*)$ where $x = \log X$ and $y = \log(Y)$.

b) Consider the difference equation $K_{t+1} = AK_t^\alpha - C_t$ where K_t is capital in period t and C_t is the consumption, respectively. Assume that the system has a steady state $(\bar{K}, \bar{C})$ with the corresponding log-transformed variables $(\bar{k}, \bar{c})$. Log-linearize both sides of the system around the steady state to obtain the approximation $k_{t+1} - \bar{k} \approx \alpha \bar{Y}(k_t - \bar{k})/\bar{K} - \bar{C}(c_t - \bar{c})/\bar{K}$, where $\bar{Y} = A\bar{K}^{\alpha}$. (Hint: remember to use the steady-state relation $\bar{K} = \bar{Y} - \bar{C}$)

4.
Consider the following nonlinear system of difference equations:

$$\begin{cases} x_{t+1} = x_t y_t + y_t^2 - 1 \\ y_{t+1} = x_t + 3x_t^2 y_t - 3 \end{cases}.$$

a) Show that $(1, 1)$ is a steady-state of the system.

b) Find a linear approximation to the right hand side of the equation (first order Taylor's approximation, or more simply the derivative of the function from $\mathbb{R}^2$ to $\mathbb{R}^2$) around the steady-state.

c) Find the eigenvalues of the linear approximation. Is the steady state locally stable? Extra: verify the eigenvalues with using Matlab's eig-function.

5.
Find the critical points of the following functions and classify them as local maxima, minima or saddle points.

a) $f(x, y) = -x^4 - y^4$.

b) $f(x, y) = x^3 + y^3$.

c) $f(x, y, z) = x^2 + 2y^2 + 3z^2 + 2xy + 2xz$.

d) $f(x) = \|Ax - b\|^2 + \delta\|x\|^2$, $\delta > 0$ (exogenous), $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m, \times n}$, and $\|\cdot\|$ is the usual norm in $\mathbb{R}^n$.

6.
Suppose that $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$ are n points in the xy-plane and suppose that we want to " fit " a straight line $y = a + bx$ to these points in such a way that the sum of the squares of the vertical deviations of the given points from the line is as small as possible. In other words, we want to choose a, b so that

$$f(a, b) = \sum_{i=1}^{n} (a + bx_i - y_i)^2$$

is as small as possible. The resulting line is called the **linear regression** line for the points $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$.

a) Find the linear regression line for the three data points

$$(-1, 2), (0, 1), (1, 0).$$

b) Find the linear regression line for the four data points

$$(-1, 2), (0, 1), (1, 0), (7, 8).$$

Graph both lines on the same set of axes and comment.

Extra: it can be shown that the solution of a problem

$$\min_{\beta} \sum_{i=1}^{m} (\beta_0 + \beta_1 x_{1i} + \ldots + \beta_n x_{ni} - y_i)^2$$

is given by $\beta = (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}y$, where $\beta = (\beta_0, \beta_1, \ldots, \beta_n)$, $y = (y_1, \ldots, y_m)$, and

$$X = \begin{pmatrix} 1 & x_{11} & \cdots & x_{n1} \\ 1 & x_{12} & \cdots & x_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{1m} & \cdots & x_{nm} \end{pmatrix}.$$

You can find the parameter vector β by using Matlab: first form X and y, then compute

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beta= inv(X'*X)*X'*y
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Alternatively you can use Matlab's optimization function fminunc or curve fitting functions (such as polyfit). Once you have found β, you can plot the regression line with Matlab (in this case $n = 1$), e.g., as follows:

```
ypoints=[beta'* [1; xmin]; beta'* [1; xmax]];
plot([xmin xmax], ypoints,'k--');
hold on;
plot(x, y,'bo');
```

where xmin is the smallest value of x and xmax is the largest. The last to lines are related to plotting the data in the same figure. Vectors x and y contain the data points (column vectors). (In statistics toolbox there is also lsline function for plotting regression lines)