4. EXERCISE

1.

a) Define the function f for all $x_1, x_2 > 0$ by

$$f(x_1, x_2) = A\left(\delta x_1^{\rho} + (1-\delta)\, x_2^{\rho}\right)^{1/\rho}, \text{ where } \delta \in [0,1], \rho \in [0,1], A > 0.$$

This is the **CES** (**C**onstant **E**lasticity of **S**ubstitution)function. For which values of ρ is this function quasiconcave?

b) Show that strictly quasiconcave functions have unique maxima (if the maxima exist).

c) Let f be an affine function on $\mathbb{R}^n$, i.e. $f(x) = Ax + b$. Show that $f^{-1}(S)$ (the pre-image of S) is a convex set if S is a convex set.

2.

Find the closed form presentation for the optimum of

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^{\mathrm{T}} Q x \quad \text{s.t. } Ax = b,$$

where $Q \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix and $A \in \mathbb{R}^{m \times n}$ has full rank, $m < n$, $n \geq 2$, $b \in \mathbb{R}^m$.

3.

Consider the following optimization problem for $\alpha > 0$:

$$\max_{x,y} \alpha x + \sqrt{y}$$

$$\begin{aligned} \text{s.t. } px + y &\leq 1, \; p > 0 \\ x &\geq 0 \\ y &\geq 0. \end{aligned}$$

i) Write down the first order necessary conditions for the problem.
ii) Find an optimum?
iii) Argue that you have found a global maximum in *ii*).

4.
A firm has a production function $Q(K,L) = 50K^2\sqrt{L}$, where K is capital, L is labor, and Q is the output. Assume that both capital and labor have a unit price 1000 and the budget of the firm is 80 000 euros.
a) Find the maximizer of Q under the budget constraint.
b) Use the Lagrange multiplier of the budget constraint to estimate how the optimal output changes as the budget is decreased by 1000 euros.
c) Compute the exact change in the optimal output when the budget is decreased by 1000 euros.

5.
Consumer's maximization problem over two-period consumption is

$$\max_{c_0,c_1}[u(c_0) + \delta u(c_1)]$$
$$\text{s.e. } c_0 \leq w_0$$
$$c_1 \leq (1+r)(w_0 - c_0)$$
$$c_0, c_1 \geq 0.$$

You may assume that at optimum $c_0, c_1 > 0$ and $c_0 < w_0$.

a) Use the first order conditions to show that at the optimum

$$\begin{aligned} MRS(c_0, c_1) &= p_0/p_1 \\ p_0c_0 + p_1c_1 &= p_0w_0, \end{aligned}$$

where p_i are the prices of consumption (figure out what these are) and MRS is the marginal rate of substitution between consumption of periods $MRS(c_0, c_1) = u'(c_0)/[\delta u'(c_1')]$.What is the Lagrange multiplier of the active constraint at optimum and what is its interpretation?

b) Under which conditions for u the solution of the first order conditions is globally optimal? What would guarantee local optimality?

6.
Consider the problem $\min_x p \cdot x$ s.t. $U(x) \geq u$, $x \geq \mathbf{0}$, where $x \in \mathbb{R}^n$ (consumption bundle), $p \in \mathbb{R}^n$ (exogenous prices), $p \gg \mathbf{0}$, and $u > 0$ is an exogenously given utility level.
a) Show that the problem is a convex optimization problem when $U(x)$ is quasiconcave.
b) Can you say something about the existence of a solution when U is continuous and increasing in all of its arguments?
c) Solve the problem by using the first order conditions when $n = 2$ and $U(x) = x_1^{\alpha}x_2^{1-\alpha}$, where $\alpha \in (0,1)$.

7. (Extra)
Assume that $u(x,a)$ is a concave and continuous function on x. Show that

$$F(a) = \arg\max_{x \in X} u(x,a)$$

is a compact and convex valued correspondence when X is a compact and convex set.